Weitzenböck formulas for Riemannian foliations

Vladimir Slesar
Faculty of Mathematics and Computer Sciences,
University of Craiova, 200585, Craiova, Romania
e-mail: vlslesar@hotmail.com

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Abstract

In this paper we study the interplay between adiabatic limits of a Riemannian foliation and the classical Weitzenböck formula. For the *leafwise* part, our study leads to a vanishing result for the first order term $\hat{E}_1$ of differential spectral sequence associated with the foliation. For the *transversal* part we obtain a Weitzenböck type formula which is an extension of the previous formula for basic forms due to Ph. Tondeur, M. Min-Oo, and E. Ruh, and is also more general than a Weitzenböck formula for transverse fiber bundle due to Y. Kordyukov.

Keywords: *Riemannian foliation; de Rham derivative; Weitzenböck formula*

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1 Introduction

Let us consider the C^∞ Riemannian foliation $\mathcal{F}$ on a closed manifold M, endowed with a *bundle-like* metric g, i.e., a metric so that the foliation is locally defined by a Riemannian submersion in a neighborhood of any point $x \in M$; as a result, the foliation has an isometric holonomy on any transverse submanifold [11]. We start out by stating some basic facts concerning *spectral* *sequence* and the *adiabatic limit* associated with the foliation. The dimension of the foliation will be denoted by p, the codimension by q and (Ω, d) will denote the de Rham complex on M. In the classical way (see e.g. [1]) we get a bigrading for Ω, induced by the foliated structure and the bundle-like metric:

$$\Omega^{u,v} = C^{\infty}\left(\bigwedge^{u} T\mathcal{F}^{\perp *} \oplus \bigwedge^{v} T\mathcal{F}^{*}\right),\ u, v \in \mathbf{Z}. \tag{1}$$

Then, the de Rham derivative and coderivative split into bihomogeneous components as follows:

$$d = d_{0,1} + d_{1,0} + d_{2,-1},\ \delta = \delta_{0,-1} + \delta_{1,0} + \delta_{-2,1}, \tag{2}$$

where the indices describe the corresponding bigrading.

Considering the space of $r-$forms of filtration degree $\geq k$:

$$\Omega_k^r = \{\omega \in \Omega^r \mid i_X \omega = 0, (\forall)\ X = X_1 \wedge ... \wedge X_{r-k+1},\ X_i \in \mathfrak{X}(\mathcal{F})\},$$

where i_X is the interior product by X, we get the following decreasing filtration:

$$\Omega := \Omega_0 \supset \Omega_1 \supset ... \supset \Omega_q \supset \Omega_{q+1} = 0,$$

which allow us to define the terms of the differential *spectral sequence* (E_k, d_k) (see e.g [2]):

$$\begin{aligned}
Z_k^{u,v} &= \Omega_u^{u+v} \cap d^{-1}(\Omega_{u+k}^{u+v+1}), \quad Z_\infty^{u,v} = \Omega_u^{u+v} \cap \ker d,\\
B_k^{u,v} &= \Omega_u^{u+v} \cap d(\Omega_{u-k}^{u+v-1}), \quad B_\infty^{u,v} = \Omega_u^{u+v} \cap \operatorname{im} d,\\
E_k^{u,v} &= \frac{Z_k^{u,v}}{Z_{k-1}^{u+1,v-1} + B_{k-1}^{u,v}}, \quad E_\infty^{u,v} = \frac{Z_\infty^{u,v}}{Z_\infty^{u+1,v-1} + B_\infty^{u,v}}.
\end{aligned}$$

In particular $Z_0^{u,v} = Z_{-1}^{u,v} = \Omega_u^{u+v}$. Let us consider $B_{-1}^{u,v} = 0$, $E_0^{u,v} = \Omega_u^{u+v}/\Omega_{u+1}^{u+v}$. Also, we get $B_u^{u,v} = B_\infty^{u,v}$ and $Z_{q-u+1}^{u,v} = Z_\infty^{u,v}$ because the filtration of Ω is of lenght $q+1$. Each homomorfism $d_k : E_k^{u,v} \to E_k^{u+k,v-k+1}$ is canonically induced by d.

The C^∞ topology on Ω induces a topology on each $E_k^{u,v}$, with respect to the bigrading. In this manner, each d_k becomes a continuous operator on $E_k = \bigoplus\limits_{u,v} E_k^{u,v}$. So we obtain two bigraded complexes: $\overline{0}_k \subset E_k$ and the quotient complex $\widehat{E}_k = E_k/\overline{0}_k$.

A differential form ω is said to be *basic* if it satisfies $i_X\omega = 0$ and $i_X d\omega = 0$ for all $X \in \mathfrak{X}(\mathcal{F})$ [8]. The link between spectral sequence terms and the *basic* cohomology $H_B^{\cdot}(\mathcal{F})$ is emphasize by the isomorphisms:

$$E_2^{\cdot,0} \cong H_B^{\cdot}(\mathcal{F}).$$

The *adiabatic limit* of a Riemannian foliation -which will represent the main ingredient in this paper, was introduced by E. Witten for a Riemannian bundle over the circle [13]. We decompose the metric $g = g_\perp \oplus g_\mathcal{F}$ with respect to the splitting $TM = T\mathcal{F}^\perp \oplus T\mathcal{F}$. Introducing a parameter $h > 0$, we define the family of metrics

$$g_h = h^{-2} g_\perp \oplus g_\mathcal{F}.$$

The limit of the Riemannian manifold (M, g_h) as $h \downarrow 0$ is known as the *adiabatic limit*, while the above parameter is known as the *adiabatic parameter*.

The starting point of our paper is represented by the joint work of Álvarez López and Kordyukov [1]. Considering the *leafwise* Laplace operator Δ_0 and its kernel $\mathcal{H}_1$, the authors proved the following Hodge-de Rham decomposition

$$\Omega = \mathcal{H}_1 \oplus \overline{\operatorname{im} d_{0,1}} \oplus \overline{\ker \delta_{0,-1}}, \tag{3}$$

which yields an isomorphism between $\mathcal{H}_1$ and $\widehat{E}_1$. This allows us to apply the classical Bochner technique (see e. g. [10]) in this special frame.

In the next section, using the above defined *adiabatic parameter* h and the bigrading we describe the terms of classical Weitzenböck formula as polynomials in h. Identifying the coefficients, this description and relation (3) will lead us in the last section to a vanishing result for $\widehat{E}_1$ in Theorem 1; this represents the corresponding vanishing result for the *leafwise* Weitzenböck formula. In the same manner we obtain in Theorem 2 a *transversal* Weitzenböck formula which turns out to be more general than the previous ones stated in [7] and [5].

2 Adiabatic limits and Riemannian foliations

The variation of Laplace operator, Levi Civita connection and metric tensor field with respect to the *adiabatic parameter* is studied in [2] and [12] . In this section we recall the main results and refine some of them using the above defined bigrading.

In the following, let us consider $\{F_a\}$, $1 \leq a \leq q$, as being C^∞ local infinitesimal transformation of $(M, \mathcal{F})$ orthogonal to the leaves, and $\{E_i\}$, $1 \leq i \leq p$, as being C^∞ local vector fields tangent to the leaves. We can assume furthermore that $\{F_a\}$ and $\{E_i\}$ are *transversal* and *leafwise* orthonormal frame fields, and as a consequence the restrictions of $\{F_a, E_i\}$ at any point

$x \in M$ where they are defined form an orthonormal basis $\{f_a, e_i\}$ in the space T_xM of local tangent vectors. Let us consider also the dual coframes $\{\theta^a, \omega^i\}$ for $\{F_a, E_i\}$, and $\{\alpha^a, \beta^i\}$ for $\{f_a, e_i\}$. The local vector fields $\{F_a\}$ will be called *basic* vector fields (see e.g [8]). For an arbitrary differential 1-form Θ, we denote by $\Theta^{\mathcal{T}}$ the transverse component and by $\Theta^{\mathcal{L}}$ the leafwise component; as a result, if ∇ is the covariant derivative and U a local tangent vector field, then we obtain the splitting $\nabla_U \Theta = (\nabla_U \Theta)^{\mathcal{T}} + (\nabla_U \Theta)^{\mathcal{L}}$. We define the transversal projection of the covariant derivative acting on differential 1-forms $\nabla_U^{\mathcal{T}} \Theta := (\nabla_U \Theta)^{\mathcal{T}}$ and the leafwise projection $\nabla_U^{\mathcal{L}} \Theta := (\nabla_U \Theta)^{\mathcal{L}}$.

Using the classical Koszul formula, we are able to express all the components of the Levi-Civita connection (determined by the transverse-leafwise decomposition) as polynomials in h. We obtain [12]:

Proposition 1. *The following equalities relate the canonical Levi-Civita connections associated to the metrics g_h and g:*

$$\begin{array}{ll} \nabla^{g_h,\mathcal{T}}_{F_a}\theta^b = \nabla^{\mathcal{T}}_{F_a}\theta^b, & \nabla^{g_h,\mathcal{L}}_{F_a}\theta^a = h^2\nabla^{\mathcal{L}}_{F_a}\theta^a, \\ \nabla^{g_h,\mathcal{T}}_{E_i}\omega^j = \nabla^{\mathcal{T}}_{E_i}\omega^j, & \nabla^{g_h,\mathcal{L}}_{E_i}\omega^j = \nabla^{\mathcal{L}}_{E_i}\omega^j, \\ \nabla^{g_h,\mathcal{T}}_{F_a}\omega^i = \nabla^{\mathcal{T}}_{F_a}\omega^i, & \nabla^{g_h,\mathcal{L}}_{F_a}\omega^i = \nabla^{\mathcal{T}}_{F_a}\omega^i, \\ \nabla^{g_h,\mathcal{T}}_{E_i}\theta^a = h^2\nabla^{\mathcal{T}}_{E_i}\theta^a, & \nabla^{g_h,\mathcal{L}}_{E_i}\theta^a = h^2\nabla^{\mathcal{L}}_{E_i}\theta^a. \end{array} \tag{4}$$

for any indices a, b, i and j, with $1 \le a, b \le q$ and $1 \le i, j \le p$, respectively.

It is possible now to express some of the curvature operator components as polynomials in h. Considering arbitrary local tangent vector fields U and V we denote by $R^{\mathcal{T}}_{g_h,U,V}$ and $R^{\mathcal{L}}_{g_h,U,V}$ the transversal and respectively the leafwise projection of the curvature operator $R_{g_h,U,V} = \nabla^{g_h}_U\nabla^{g_h}_V - \nabla^{g_h}_V\nabla^{g_h}_U - \nabla^{g_h}_{[U,V]}$ acting on differential forms; also we denote by $R^{\perp}$ the *transversal curvature operator* (see e.g. [6]) and by $R^{\mathcal{F}}$ the *leafwise curvature operator* (see [3]).

Proposition 2. *The components of the curvature operator associated to the metrics g_h and g are subject to the relations [12]:*

$$\begin{array}{rcl} R^{\mathcal{T}}_{g_h,f_a,f_b}\alpha^c &=& R^{\perp}_{f_a,f_b}\theta^c + h^2 R^{\mathcal{T},2}_{f_a,f_b}\theta^c, \\ R^{\mathcal{L}}_{g_h,e_i,e_j}\beta^k &=& R^{\mathcal{F}}_{e_i,e_j}\beta^k + h^2 R^{\mathcal{L},2}_{e_i,e_j}\beta^k. \\ R^{\mathcal{L}}_{g_h,e_i,f_a}\alpha^c &=& h^2 R^{\mathcal{L},2}_{e_i,f_a}\alpha^c + h^4 R^{\mathcal{L},4}_{e_i,f_a}\alpha^c, \\ R^{\mathcal{T}}_{g_h,e_i,e_j}\alpha^c &=& h^4 R^{\mathcal{T},4}_{e_i,e_j}\alpha^c + h^2 R^{\mathcal{T},2}_{e_i,e_j}\alpha^c. \\ R^{\mathcal{L}}_{g_h,f_a,f_b}\alpha^c &=& h^2 R^{\mathcal{L},2}_{f_a,f_b}\alpha^c, \\ R^{\mathcal{T}}_{g_h,e_i,f_a}\alpha^c &=& h^2 R^{\mathcal{T},2}_{e_i,f_a}\alpha^c. \end{array} \tag{5}$$

In the following let us now consider the classical Weitzenböck formula (see e.g. [10]). We take an orthonormal frame field $\{\mathcal{E}_i\}$ in the neighborhood of an arbitrary point $x \in M$; the restriction of $\{\mathcal{E}_i\}$ at x induces an orthonormal basis $\{\epsilon_i\}$ at x such that $\nabla_{\epsilon_i}\mathcal{E}_j = 0$, with $1 \leq i, j \leq n$. If $\{\Theta^i\}$ and $\{\theta^i\}$ are the dual coframes for $\{\mathcal{E}_i\}$ and $\{\epsilon_i\}$ respectively, considering that $d = \sum_i \Theta^i \wedge \nabla_{\mathcal{E}_i}$ and $\delta = -\sum_i i_{\mathcal{E}_i}\nabla_{\mathcal{E}_i}$, we can express the Laplace operator:

$$\Delta = d\delta + \delta d = \nabla^*\nabla + K, \tag{6}$$

where $K =: \sum_{i<j} \theta^i \cdot \theta^j \cdot R_{\epsilon_i,\epsilon_j}$, and $\theta \cdot \omega := \theta \wedge \omega - i_{\theta^\sharp}\omega$, for any 1-form θ and arbitrary form ω, the tangent vector $\theta^\sharp$ being determined by the relation $\langle \theta^\sharp, v\rangle = \theta(v)$, for any $v \in T_xM$.

In the introductory section we mentioned the canonical splitting $TM = T\mathcal{F}^\perp \oplus T\mathcal{F}$. We have also a splitting of the cotangent bundle $TM^* = T\mathcal{F}^{\perp *} \oplus T\mathcal{F}^*$. The canonical transversal and leafwise projection operator will be denoted by $\mathrm{pr}^{\mathcal{T}}$ and $\mathrm{pr}^{\mathcal{L}}$ respectively. We can consider the *rescaling homomorphism* $\Theta_h : (TM^*, g_h) \to (TM^*, g)$, defined using the identity operators $\mathrm{id}_{T\mathcal{F}^{\perp *}}$ and $\mathrm{id}_{T\mathcal{F}^*}$:

$$\Theta_h = h\,\mathrm{id}_{T\mathcal{F}^{\perp *}} \oplus \mathrm{id}_{T\mathcal{F}^*}. \tag{7}$$

The induced *rescaling homomorphism* on differential forms or tensor fields will be denoted also by Θ_h. One can prove that these are in fact isometries of Riemannian vector bundles (see e.g [6]). This allow us to define the *rescaled* operators $\Delta_h := \Theta_h \Delta_{g_h} \Theta_h^{-1}$, $\nabla^h := \Theta_h \nabla^{g_h} \Theta_h^{-1}$ and $K^h := \Theta_h K_{g_h} \Theta_h^{-1}$.

Using the above Riemannian vector bundles isometry and applying (6) for $\Theta_h^{-1}\omega$, we obtain the formula

$$\langle \Delta_h\omega, \omega\rangle = \left\langle \nabla^h\omega, \nabla^h\omega\right\rangle + \left\langle K^h\omega, \omega\right\rangle, \tag{8}$$

where the inner product is obtained integrating on the closed Riemannian manifold M (see e.g. [4]).

We will express all terms of (8) as polynomials in h. The first term is studied in [2]:

$$\begin{aligned}\langle \Delta_h\omega, \omega\rangle \;=\; & \langle \Delta_0\omega, \omega\rangle + h\,\langle (D_\perp D_0 + D_0 D_\perp)\omega, \omega\rangle \\ & + h^2\left(\langle (D_0F + FD_0)\omega, \omega\rangle + \langle \Delta_\perp\omega, \omega\rangle\right) \\ & + h^3\,\langle (D_\perp F + FD_\perp)\omega, \omega\rangle + h^4\left\langle F^2\omega, \omega\right\rangle,\end{aligned} \tag{9}$$

where F is the 0-th order operator $d_{2,-1} + \delta_{-2,1}$, $D_0 := d_{0,1} + \delta_{0,-1}$ and $\Delta_0 := D_0^2$ are the *leafwise* Dirac and Laplace operators, while $D_\perp := d_{1,0} + \delta_{-1,0}$ and $\Delta_\perp := d_{1,0}\delta_{-1,0} + \delta_{-1,0}d_{1,0}$ represent the *transversal* Dirac and Laplace operators, respectively .

In order to obtain a *transversal* Weitzenböck formula we need a refinement of (9) in the presence of the bigrading induced by the foliated structure and the bundle-like metric. We state now the following result:

Proposition 3. *If we take $\omega = \omega^{u,v}$, then the coefficient of h^2 of the above polynomial becomes $\langle \Delta_\perp \omega^{u,v}, \omega^{u,v} \rangle$.*

Proof. It is easy to see that

$$\begin{aligned}(D_0F + FD_0)\omega^{u,v} \quad = \quad & (d_{0,1}d_{2,-1} + d_{2,-1}d_{0,1})\,\omega^{u,v} \\ & + (\delta_{0,-1}d_{2,-1} + d_{2,-1}\delta_{0,-1})\,\omega^{u,v} \\ & + (\delta_{-2,1}d_{0,1} + d_{0,1}\delta_{-2,1})\,\omega^{u,v} \\ & + (\delta_{0,-1},\delta_{-2,1} + \delta_{-2,1}\delta_{0,-1})\,\omega^{u,v},\end{aligned}$$

where $(d_{0,1}d_{2,-1} + d_{2,-1}d_{0,1})\,\omega^{u,v} \in \Omega^{u+2,v}$, $(\delta_{0,-1}d_{2,-1} + d_{2,-1}\delta_{0,-1})\,\omega^{u,v} \in \Omega^{u+2,v-2}$, $(\delta_{-2,1}d_{0,1} + d_{0,1}\delta_{-2,1})\,\omega^{u,v} \in \Omega^{u-2,v+2}$, and finally $(\delta_{0,-1}\delta_{-2,1} + \delta_{-2,1}\delta_{0,-1})\,\omega^{u,v} \in \Omega^{u-2,v}$; as a result, the term $\langle (D_0F + FD_0)\omega^{u,v}, \omega^{u,v} \rangle$ vanishes. $\square$

For the second term we consider the covariant derivative ∇ induced on $\Omega^{u,v}$, with u and v satisfying $u + v = r$:

$$\nabla : \Omega^{u,v} \longrightarrow C^\infty(TM^*) \otimes C^\infty(\Lambda^r TM^*).$$

We refine the covariant derivative in the presence of the canonical projections operators $\mathrm{pr}^{\mathcal{T}}$, $\mathrm{pr}^{\mathcal{L}}$-determined by the foliated structure, and the canonical projections $\pi_{u,v}$, $\pi_{u-1,v+1}$ and $\pi_{u+1,v-1}$-induced by the bigrading, defining the following six differential operators:

$$\begin{array}{ll} \nabla_{\mathcal{T},0,0} = (\mathrm{pr}^{\mathcal{T}} \otimes \pi_{u,v}) \circ \nabla, & \nabla_{\mathcal{L},0,0} = (\mathrm{pr}^{\mathcal{L}} \otimes \pi_{u,v}) \circ \nabla, \\ \nabla_{\mathcal{T},-1,1} = (\mathrm{pr}^{\mathcal{T}} \otimes \pi_{u-1,v+1}) \circ \nabla, & \nabla_{\mathcal{L},-1,1} = (\mathrm{pr}^{\mathcal{L}} \otimes \pi_{u-1,v+1}) \circ \nabla, \\ \nabla_{\mathcal{T},1,-1} = (\mathrm{pr}^{\mathcal{T}} \otimes \pi_{u+1,v-1}) \circ \nabla, & \nabla_{\mathcal{L},1,-1} = (\mathrm{pr}^{\mathcal{L}} \otimes \pi_{u+1,v-1}) \circ \nabla. \end{array}$$

The above operators can be naturally extended from $\Omega^{u,v}$ to Ω.

Remark 1. *Considering the rescaled covariant derivative, it follows easily that:*

$$\nabla^h = \nabla^h_{\mathcal{T},0,0} + \nabla^h_{\mathcal{L},0,0} + \nabla^h_{\mathcal{T},-1,1} + \nabla^h_{\mathcal{L},-1,1} + \nabla^h_{\mathcal{T},1,-1} + \nabla^h_{\mathcal{L},1,-1}. \tag{10}$$

Following [1], we choose a foliated chart $\mathcal{U}$ on M; then we obtain the following description of the de Rham complex:

$$\Omega^{u,v}(\mathcal{U}) = \Omega^u(\mathcal{U}/\mathcal{F}_{\mathcal{U}}) \wedge \Omega^{0,v}(\mathcal{U}) \equiv \Omega^u(\mathcal{U}/\mathcal{F}_{\mathcal{U}}) \otimes \Omega^{0,v}(\mathcal{U}), \tag{11}$$

As a consequence, we take $\alpha \in \Omega^u(\mathcal{U}/\mathcal{F}_{\mathcal{U}})$ and $\beta \in \Omega^{0,v}(\mathcal{U})$, and we evaluate the above operators acting on differential forms of the type $\alpha\wedge\beta$, the general formula being easy to obtain by linearity. Considering also the change of the bigrading, we write all the operators only using the Levi-Civita connection associated to g and the adiabatic parameter h:

$$\begin{aligned}
&\nabla^h_{\mathcal{T},0,0}(\alpha\wedge\beta) = h\left(\nabla^{g_h}_{\mathcal{T},0,0}\alpha\otimes\beta + \alpha\otimes\nabla^{g_h}_{\mathcal{T},0,0}\beta\right) = h\nabla_{\mathcal{T},0,0}(\alpha\wedge\beta),\\
&\nabla^h_{\mathcal{L},0,0}(\alpha\wedge\beta) = \nabla^{g_h}_{\mathcal{L},0,0}(\alpha\wedge\beta) = \alpha\otimes\nabla_{\mathcal{L},0,0}\beta + h^2\,\nabla_{\mathcal{L},0,0}\alpha\otimes\beta,\\
&\nabla^h_{\mathcal{T},-1,1}(\alpha\wedge\beta) = \nabla^{g_h}_{\mathcal{T},-1,1}(\alpha\wedge\beta) = h^2\nabla_{\mathcal{T},-1,1}\alpha\wedge\beta = h^2\nabla_{\mathcal{T},-1,1}(\alpha\wedge\beta),\\
&\nabla^h_{\mathcal{L},-1,1}(\alpha\wedge\beta) = h^{-1}\nabla^{g_h}_{\mathcal{L},-1,1}(\alpha\wedge\beta) = h\nabla_{\mathcal{L},-1,1}\alpha\wedge\beta = h\nabla_{\mathcal{L},-1,1}(\alpha\wedge\beta),\\
&\nabla^h_{\mathcal{T},1,-1}(\alpha\wedge\beta) = h^2\nabla^{g_h}_{\mathcal{T},1,-1}(\alpha\wedge\beta) = h^2\,\alpha\wedge\nabla_{\mathcal{T},1,-1}\beta = h^2\nabla_{\mathcal{T},1,-1}(\alpha\wedge\beta),\\
&\nabla^h_{\mathcal{L},1,-1}(\alpha\wedge\beta) = h\,\nabla^{g_h}_{\mathcal{L},1,-1}(\alpha\wedge\beta) = h\,\alpha\wedge\nabla_{\mathcal{L},1,-1}\beta = h\,\nabla_{\mathcal{L},1,-1}(\alpha\wedge\beta).
\end{aligned}$$

In the following we denote $\mathrm{id}_{\Omega^u(\mathcal{U}/\mathcal{F}_{\mathcal{U}})}\otimes\nabla_{\mathcal{L},0,0}$ by $\nabla^0_{\mathcal{L},0,0}$ and $\nabla_{\mathcal{L},0,0}\otimes\mathrm{id}_{\Omega^{0,\cdot}(\mathcal{U})}$ by $\nabla^2_{\mathcal{L},0,0}$. Consequently, we obtain the following result [12]:

Proposition 4. *The following polynomial description is valid:*

$$\begin{aligned}
\|\nabla^h\omega\|^2 \;=\; & \|\nabla^0_{\mathcal{L},0,0}\omega\|^2 + 2h\left(\left\langle\nabla^0_{\mathcal{L},0,0}\omega, \nabla_{\mathcal{L},1,-1}\omega\right\rangle + \left\langle\nabla^0_{\mathcal{L},0,0}\omega, \nabla_{\mathcal{L},-1,1}\omega\right\rangle\right)\\
&+h^2\left(2\left\langle\nabla^0_{\mathcal{L},0,0}\omega, \nabla^2_{\mathcal{L},0,0}\omega\right\rangle + 2\left\langle\nabla_{\mathcal{L},1,-1}\omega, \nabla_{\mathcal{L},-1,1}\omega\right\rangle + \|\nabla_{\mathcal{T},0,0}\omega\|^2\right.\\
&\left.+\|\nabla_{\mathcal{L},1,-1}\omega\|^2 + \|\nabla_{\mathcal{L},-1,1}\omega\|^2\right) + o(h^2).
\end{aligned}$$

The following corollary will be useful in the next section:

Corollary 1. *If $\omega = \omega^{u,v}$, then the coefficient of h^2 in the above polynomial description becomes*

$$2\left\langle\nabla^0_{\mathcal{L},0,0}\omega^{u,v}, \nabla^2_{\mathcal{L},0,0}\omega^{u,v}\right\rangle + \|\nabla_{\mathcal{T},0,0}\omega^{u,v}\|^2 + \|\nabla_{\mathcal{L},1,-1}\omega^{u,v}\|^2 + \|\nabla_{\mathcal{L},-1,1}\omega^{u,v}\|^2.$$

Proof. If we take $\omega = \omega^{u,v}$, then $\nabla_{\mathcal{L},1,-1}\omega^{u,v} \in \Omega^{u+1,v-1}$ and $\nabla_{\mathcal{L},-1,1}\omega^{u,v} \in \Omega^{u-1,v+1}$, so $\langle\nabla_{\mathcal{L},1,-1}\omega^{u,v}, \nabla_{\mathcal{L},-1,1}\omega^{u,v}\rangle = 0$, and the conclusion follows. □

In order to investigate the last term of (8), let us observe that the formulas (5) allow us to express the curvature components as polynomials in h [12]:

$$K^h = \sum_{i=0}^{4} h^i \cdot K^i$$

and in accordance with the bigrading, that means:

$$K^i = K^i_{-2,2} + K^i_{-1,1} + K^i_{0,0} + K^i_{1,-1} + K^i_{2,-2} \tag{12}$$

for $0 \leq i \leq 4$.

As a consequence, we end this section with the following result:

Proposition 5. *If $\omega = \omega^{u,v}$, then the coefficient of h^2 in the last term of (8) is $\left\langle K^2_{0,0}\omega^{u,v}, \omega^{u,v}\right\rangle$.*

Proof. If we make $i = 2$, then we obtain the result observing that the terms $\left\langle K^2_{-2,2}\omega^{u,v}, \omega^{u,v}\right\rangle$, $\left\langle K^2_{-1,1}\omega^{u,v}, \omega^{u,v}\right\rangle$, $\left\langle K^2_{1,-1}\omega^{u,v}, \omega^{u,v}\right\rangle$ and $\left\langle K^2_{2,-2}\omega^{u,v}, \omega^{u,v}\right\rangle$ vanish. $\square$

3 A leafwise and a transversal Weitzenböck formula

In [12], the relation (8) is considered for $\omega^{u,v} \in \Omega^{u,v}$. Writing the both sides of the equality as polynomials in h and considering only the coefficients of h^0, a *leafwise* Weitzenböck formula is obtained:

$$\langle \Delta_0\omega^{u,v}, \omega^{u,v}\rangle = \|\nabla^0_{\mathcal{L},0,0}\omega^{u,v}\|^2 + \left\langle K^0_{0,0}\omega^{u,v}, \omega^{u,v}\right\rangle . \tag{13}$$

where $K^0_{0,0}(\alpha \wedge \beta) = \alpha \otimes \sum\limits_{i<j} \beta^i \cdot \beta^j \cdot R^{\mathcal{F}}_{e_i,e_j}\beta$. As a consequence, we obtain now the following vanishing result:

Theorem 1. *If the curvature operator along the leaves $R^{\mathcal{F}}$ is positive at every point, then $\widehat{E}^{u,v}_1 = 0$ for $v = 1, 2, .., p-1$. Moreover, if the foliation has dense leaves, $R^{\mathcal{F}}$ is nonnegative and positive at some point, then we get the same result.*

Proof. As in [1], we denote $\ker \Delta_0$ by $\mathcal{H}_1$. Let $\omega^{u,v}$ be a differential form in $\mathcal{H}^{u,v}$, for $v = 1, 2, .., p-1$. At an arbitrary point $x \in M$, we consider first $\omega^{u,v} = \alpha \wedge \beta$, the general case being easy to obtain by linearity. Arguing as in the classical case (see e.g., [10]), we get:

$$\left\langle K^0_{0,0}(\alpha \wedge \beta), \alpha \wedge \beta\right\rangle_x$$

$$
\begin{aligned}
&= \|\alpha\|_x^2 \cdot \left\langle \sum_{i<j} \beta^i \cdot \beta^j \cdot R^{\mathcal{F}}_{e_i,e_j}\beta, \beta \right\rangle_x \\
&= \frac{1}{2}\|\alpha\|_x^2 \cdot \left\langle \sum_{i<j} \left[\beta^i \cdot \beta^j, R^{\mathcal{F}}_{e_i,e_j}\beta\right], \beta \right\rangle_x \\
&= -\frac{1}{2}\|\alpha\|_x^2 \cdot \left\langle \sum_{i<j} R^{\mathcal{F}}_{e_i,e_j}\beta, \left[\beta^i \cdot \beta^j, \beta\right] \right\rangle_x \\
&= -\frac{1}{4}\|\alpha\|_x^2 \cdot \left\langle \sum_{i<j,k<l} \left\langle R^{\mathcal{F}}_{e_i,e_j}e_k, e_l \right\rangle_x \left[\beta^k \cdot \beta^l, \beta\right], \left[\beta^i \cdot \beta^j, \beta\right] \right\rangle_x \\
&= \frac{1}{4}\|\alpha\|_x^2 \cdot \sum_{i<j,k<l} \left\langle R^{\mathcal{F}}(e_i \wedge e_j), e_k \wedge e_l \right\rangle_x \left\langle \left[\beta^k \cdot \beta^l, \beta\right], \left[\beta^i \cdot \beta^j, \beta\right] \right\rangle_x \quad (14)
\end{aligned}
$$

Now observe that the elements $\{e_i \wedge e_j\}_{i<j}$ form an orthonormal basis for $\bigwedge^2 T_x\mathcal{F}$, and the $\{\beta^i \wedge \beta^j\}_{i<j}$ are the dual basis for $\bigwedge^2 T_x^*\mathcal{F}$. The last expression is clearly independent of the choise of orthonormal bases $\{f_a, e_i\}$ and $\{\alpha^a, \beta^i\}$. We select the orthonormal basis $\{\Xi_s\}_{1\leq s\leq \frac{p(p-1)}{2}}$ for $\bigwedge^2 T_x\mathcal{F}$ such that $R^{\mathcal{F}}(\Xi_s) = \lambda_s \Xi_s$. Considering $\{\Phi^s\}_{1\leq s\leq \frac{p(p-1)}{2}}$ the dual basis for $\bigwedge^2 T_x^*\mathcal{F}$, we obtain:

$$
\left\langle K^0_{0,0}(\alpha\wedge\beta), \alpha\wedge\beta \right\rangle_x = \frac{1}{4}\sum_s \lambda_s \|[\Phi^s, \beta]\|_x^2 \cdot \|\alpha\|_x^2 .
$$

If $d_{0,1}\omega^{u,v} = \delta_{0,-1}\omega^{u,v} = 0$, then

$$
\|\nabla^0_{\mathcal{L},0,0}\omega^{u,v}\|^2 + \frac{1}{4}\int_M \sum_s \lambda_s \|[\Phi^s, \beta]\|_x^2 \cdot \|\alpha\|_x^2 \, d\mu_x = 0.
$$

As both terms are nonnegative, they both vanish, so $\nabla^0_{\mathcal{L},0,0}\omega^{u,v} = \alpha \otimes \nabla_{\mathcal{L},0,0}\beta = 0$ and $\lambda_s \|[\Phi^s, \beta]\|_x^2 \cdot \|\alpha\|_x^2 = 0$. If $\lambda_s = 0$ for all s, the only way this can happen is if $\alpha = 0$, or $[\Phi^s, \beta] = 0$ for all s. The last condition is known to be equivalent to $\beta = 0$ [10], so the both situations imply $\omega^{u,v} = 0$. Now, suppose the foliation has dense leaves. If $R^{\mathcal{F}} \geq 0$ and $R^{\mathcal{F}} > 0$ at some point, than we have $\alpha = 0$ or $\beta = 0$ at the point. As α is a basic form and $\nabla_{\mathcal{L},0,0}\beta = 0$ imply that β is parallel along the leaves, we obtain in all cases that $\omega^{u,v} = 0$. By consequence, $\mathcal{H}^{u,v}_1 = 0$, for $v = 1, 2, .., p-1$. As pointed out in the introductory section, starting with the Hodge decomposition (3) and arguing as in the classical case, it was proved in [2] that $\widehat{E}_1$ and $\mathcal{H}_1$ are isomorphic topological vector spaces. Considering now the induced bigrading, we get $\widehat{E}_1^{u,v} \cong \mathcal{H}_1^{u,v}$, so $\widehat{E}_1^{u,v} = 0$ for $v = 1, 2, .., p-1$. $\square$

Remark 2. *For a* leafwise *Weitzenböck-Lichnerowicz formula and for a corresponding vanishing result in the special case of a foliation endowed with a spin structure, see [9, Appendix C].*

Using now the same technique as in [12], we obtain a *transversal* Weitzenböck formula:

Theorem 2. *If $\omega \in \Omega^r$ is a differential form of degree r defined on M, then the following relation holds:*

$$\begin{aligned} \langle \Delta_\perp \omega, \omega \rangle &= 2 \left\langle \nabla^0_{\mathcal{L},0,0}\omega, \nabla^2_{\mathcal{L},0,0}\omega \right\rangle + \left\| \nabla_{\mathcal{T},0,0}\omega \right\|^2 + \left\| \nabla_{\mathcal{L},1,-1}\omega \right\|^2 \qquad (15) \\ &\quad + \left\| \nabla_{\mathcal{L},-1,1}\omega \right\|^2 + \left\langle K^2_{0,0}\omega, \omega \right\rangle . \end{aligned}$$

Proof. We collect the coefficients of h^2 in (8) in accordance with Proposition 3, Proposition 5 and Corollary 1; by linearity, we end up with the above general *transversal* Weitzenböck formula. $\square$

Remark 3. *The above formula is more general than the Weitzenböck formula presented in [7] which works for basic forms and also more general than* transverse *Weitzenböck type formula in [5, Theorem 8] which works on transverse fiber bundle.*

Remark 4. *In [7], a Weitzenböck type formula for the* transversal *Laplacian allow the authors to obtain vanishing results concerning the basic de Rham complex of a Riemannian foliation. In certain situations, a useful tool for studying basic de Rham complex is the associated spectral sequence(see e.g [2]). The spectral sequence terms do not contain only basic differential forms, so our* transversal *Weitzenböck type formula written for differential forms of arbitrary degree might help us, at least in some particular cases, to investigate the cohomology of a Riemannian foliation.*

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